Further Calculus V Cheat Sheet (A Level Only)

Arc Length of a Curve

Arc Length of a Curve Expressed in Cartesian Coordinates

Integration is used to find the arc length of a curve between two end points. Consider a differentiable curve $y = f(x)$, given in cartesian coordinates. To find the length of the arc of the curve between the points $x = a$ and $x = b$, the following formula is used:

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

This formula is given in the formula booklet.

Example 1: Find the exact length of the arc of the curve $y = 4\cosh\left(\frac{x}{4}\right)$ between the points $x = 0$ and $x = 8$.

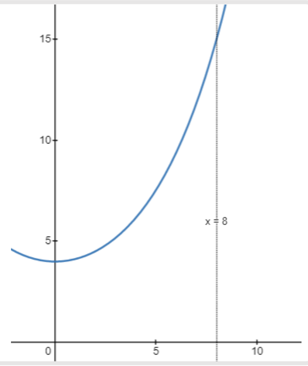
Find the derivative of $y = 4\cosh\left(\frac{x}{4}\right)$, using the chain rule. Then find and simplify $1 + \left(\frac{dy}{dx}\right)^2$ using the hyperbolic identity $\cosh^2(x) - \sinh^2(x) \equiv 1$. Take just the positive square root as for $x \in [0,8]$, $\cosh\left(\frac{x}{4}\right) > 0$. Then evaluate the integral at each limit, remembering that $\sinh(0) = 0$. See the final column for the graph of this function, including the upper limit, $x = 8$.

$$\frac{dy}{dx} = 4 \cdot \frac{1}{4} \cdot \sinh\left(\frac{x}{4}\right) = \sinh\left(\frac{x}{4}\right)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\left(\frac{x}{4}\right) = \cosh^2\left(\frac{x}{4}\right)$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \cosh\left(\frac{x}{4}\right)$$

$$\therefore s = \int_0^8 \cosh\left(\frac{x}{4}\right) dx = \left[4\sinh\left(\frac{x}{4}\right)\right]_0^8 = 4\sinh(2)$$



Arc Lengths of a Curve Expressed in Parametric Coordinates

For curves given by parametric equations, there is another formula, once again given in the formula booklet. For a curve defined via $x = x(t), y = y(t)$, the length of the arc length of the portion of the curve between the values $t = t_1$ and $t = t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$$

Example 2: Find the exact arc length of the curve given by parametric equations $x = \sin^3(t), y = \cos^3(t)$ between $t = 0$ and $t = \frac{\pi}{2}$.

Differentiate $x = x(t)$ and $y = y(t)$ with respect to t. Square and sum these derivatives to obtain an expression for:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

Simplify by factorising out;

$$9\sin^2(t)\cos^2(t)$$

and using the identity;

$$\sin^2(x) + \cos^2(x) \equiv 1$$

and the double angle formula;

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$x = \sin^3(t)\,, \frac{dx}{dt} = 3\sin^2(t)\cos(t)$$

$$y = \cos^3(t)\,, \frac{dy}{dt} = 3\cos^2(t)\sin(t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^4(t)\cos^2(t) + 9\cos^4(t)\sin^2(t)$$

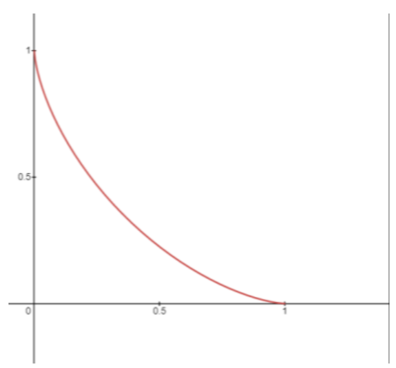
$$= 9\sin^2(t)\cos^2(t)(\sin^2(t) + \cos^2(t)) = 9\sin^2(t)\cos^2(t)$$

$$= 9\left(\frac{\sin(2t)}{2}\right)^2 = \left(\frac{3}{2}\sin(2t)\right)^2$$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{3}{2}\sin(2t)$$

Evaluate the integral between the two limits for t to find the arc length. The graph of this function is in the graph on the right.

$$\int_0^{\frac{\pi}{2}} \frac{3}{2}\sin(2t)\, dt = -\frac{3}{4}[\cos(2t)]_0^{\frac{\pi}{2}} = -\frac{3}{4}((-1) - 1) = \frac{3}{2}$$



Area of a Surface of Revolution

Area of a Surface of Revolution in Cartesian Coordinates

Rotating a curve around either axis generates a solid of revolution. For a curve defined between points $x = a$ and $x = b$, and rotated a full turn around the x-axis, the following integral formula, given in the formula book, is used to find the area of a surface of revolution:

$$S_x = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

This formula will give the curved surface area of the solid but does not include the area of any bases that the solid may have.

Example 3: Find the surface area of the cone of height 12cm and base radius 9cm by modelling the cone as a surface of revolution.

Model the cone with a straight line revolved around the x-axis with the vertex at the origin. Since it has a height of 12 and base radius of 9, the straight line connecting the vertex to the base must pass through the point (12,9). So, the straight line is given by the equation $y = \frac{9}{12}x$. Find the derivative of this function, and then use this to find:

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

As the height of the cone is 12 and the vertex is at the origin, integrate between $x = 0$ and $x = 12$. Evaluating the integral will then give the curved surface area of this cone. Add this to the area of circular base to obtain the total surface area of the cone.

$$y = \frac{9}{12}x = \frac{3}{4}x.\ \frac{dy}{dx} = \frac{3}{4} \Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

$$\therefore S_x = \int_0^{12} 2\pi \cdot \frac{3}{4}x \cdot \frac{5}{4}\, dx = \frac{15\pi}{8}\int_0^{12} x\, dx = \frac{15\pi}{16}[x^2]_0^{12}$$

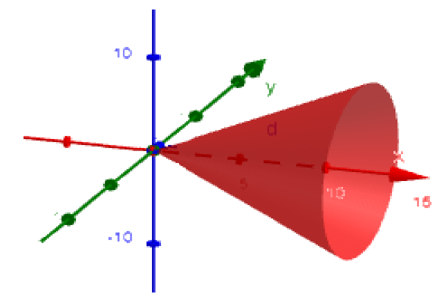
$$= 135\pi \text{ cm}^2.$$

Area of the circular base:

$$\pi r^2 = \pi(9)^2 = 81\pi$$

$\therefore$ Total surface area is

$$81\pi + 135\pi = 216\pi \text{ cm}^2.$$



Surface Area of Revolution in Parametric Coordinates

For curves defined parametrically, there is another formula for the curved surface area of revolution. For the revolution of a curve defined by $x = x(t), y = y(t)$ between the points $t = t_1$ and $t = t_2$, the surface area of revolution is given by

$$S_x = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 4: A curve is defined by the parametric equations $x = 2\sinh(t)\,, y = \frac{1}{2}\cosh(2t)$. The arc of this curve between $t = 0$ and $t = ln(2)$ is rotated 2π radians about the x-axis. Find the area of the resulting surface of revolution.

Begin by differentiating $x = x(t),\ y = y(t)$. Find and simplify an expression for;

$$y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

using a double angle formula for hyperbolic trigonometric functions and by using a hyperbolic identity. Use the exponential definition of $\cosh(t)$ to integrate between the given limits. The solid of revolution is shown in the graph on the right.

$$\frac{dx}{dt} = 2\cosh(t)\,, \frac{dy}{dt} = \sinh(2t) \ \Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\cosh^2(t) + \sinh^2(2t)$$

$$\sinh(2t) = 2\sinh(t)\cosh(t),\ \cosh^2(t) = 1 + \sinh^2(t) \Rightarrow$$

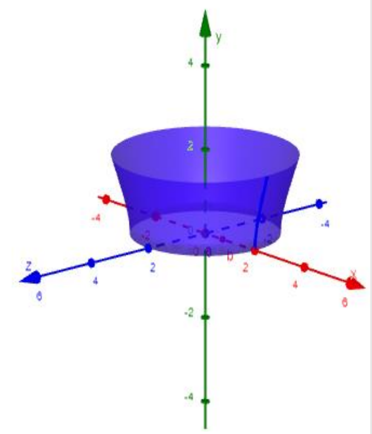
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\cosh^2(t)(1 + \sinh^2(t)) = 4\cosh^4(t)$$

$$\therefore y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{1}{2}\cosh(2t)\sqrt{4\cosh^4(t)} = \cosh(2t)\cosh^2(t)$$

$$\cosh(2t)\cosh^2(t) = \frac{1}{8}(e^x + e^{-x})^2(e^{2x} + e^{-2x}) = \frac{1}{8}(e^{4x} + 2e^{2x} + 2e^{-2x} + e^{-4x} + 2)$$

$$\therefore S_x = 2\pi\int_0^{ln(2)} \cosh^2(t)\cosh(2t)\, dt = \frac{\pi}{4}\int_0^{ln(2)} e^{4x} + 2e^{2x} + 2e^{-2x} + e^{-4x} + 2\, dt$$

$$= \frac{\pi}{4}\left[\frac{e^{4x}}{4} + e^{2x} - e^{-2x} - \frac{e^{-4x}}{4} + 2x\right]_0^{ln(2)} = \frac{\pi(495 + 128\,ln(2))}{256}$$



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